Quiz 1
Business Mathematics

31^{st} January 2006

Time: 1 hour (10–11pm)

1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

Find all the eigenvalues and a basis for each eigenspace. If possible, find invertible matrices P such that $P^{-1}AP$ is diagonal.

Solution. Form the characteristic matrix,

$$tI - A = \begin{bmatrix} t-3 & -1 & -1 \\ -2 & t-4 & -2 \\ -1 & -1 & t-2 \end{bmatrix}$$

Then,

$$\begin{aligned} |\, tI - A \,| &= \begin{vmatrix} t-3 & -1 & -1 \\ -2 & t-4 & -2 \\ -1 & -1 & t-3 \end{vmatrix} \\ &= (t-3)\left((t-4)(t-3)-2\right) + \left((t-3)(-2)-2\right) - (2+t-4) \\ &= t^3 - 10t^2 + 28t - 24 \\ &= (t-2)\left(t^2 - 8t + 12\right) \\ &= (t-2)^2(t-6) \end{aligned}$$

Therefore the eigenvalues are 2 and 6.

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For the eigenvalue 2;

$$\begin{pmatrix} 2-3 & -1 & -1 \\ -2 & 2-4 & -2 \\ -1 & -1 & 2-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives $x+y+z = 0$. The system has thus two free variables, for example $\{x = 1, y = 0, z = -1\}$ and $\{x = 1, y = -1, z = 0\}$. Let $u = (1, 0, -1)$ and $v = (1, -1, 0)$. Then all linear combinations of u and v forms a set of all possible eigenvectors, which together with zero vector comprise the eigenspace corresponding to the eigenvalue 2. In other words, u and v form a basis of the eigenspace of the eigenvalue 2.

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For the eigenvalue 6;

$$\begin{pmatrix} 6-3 & -1 & -1 \\ -2 & 6-4 & -2 \\ -1 & -1 & 6-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use Gaussian elimination to find the solution of this system of equation. First form an augmented matrix

$$\begin{pmatrix} 3 & -1 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix}$$

$(I) \leftrightarrow (III)$, $(3 \quad -1 \quad -1 \quad 0) \leftrightarrow (-1 \quad -1 \quad 3 \quad 0)$;

$$\begin{pmatrix} -1 & -2 & 3 & 0 \\ -2 & 2 & -2 & 0 \\ 3 & -1 & -1 & 0 \end{pmatrix}$$

$-1(I)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix}$$

$(II)+2(I)$, $(-2 \quad 2 \quad -2 \quad 0)+2(1 \quad 1 \quad -3 \quad 0)$; $(III)-3(I)$, $(3 \quad -1 \quad -1 \quad 0)-3(1 \quad 1 \quad -3 \quad 0)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & -4 & -8 & 0 \end{pmatrix}$$

$\frac{1}{4}(II)$, $(1 \quad -2 \quad 0)$; $(III)+4(II)$, $(-4 \quad -8 \quad 0)+4(1 \quad -2 \quad 0)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore rank of A is 2. The system has only one free variable, that is to say, one independent solution. Any particular non-zero solution generates its solution space, that is the eigenspace, for example $x = 1$, $y = 2$ and $z = 1$. So $w = (1, 2, 1)$ forms a basis of the eigenspace of the eigenvalue 6.

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Let P be $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ Since $\det P = 1(-1) - 1(2+1) = -4 \neq 0$, hence P has an inverse. Therefore P is the required matrix such that $P^{-1}AP$ is diagonal.

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Bibliography

Seymour Lipschutz. *Theory and problems of Linear Algebra.* Schaum's Outline Series, McGraw-Hill, 1987(1968x)